

Non-relativistic contributions in order $\alpha^5 m_\mu c^2$ to the Lamb shift in muonic hydrogen, deuterium and helium ion

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Contributions to the energy levels in light muonic atoms and, in particular, to the Lamb shift fall into a few well-distinguished classes. The related diagrams are calculated using different approaches. In particular, there is a specific kind of non-relativistic contributions. Here we consider such corrections to the Lamb shift in order $\alpha^5 m_\mu$. These contributions are due to free vacuum polarization loops as well as to various effects of light-by-light scattering. The closed loop in the related diagrams is an electronic one, which allows a non-relativistic consideration of the muon. Both kinds of contributions have been known for a while, however, the results obtained up to date are only partial ones. We complete a calculation of the $\alpha^5 m_\mu$ contributions for muonic hydrogen. The results are also adjusted for muonic deuterium and muonic helium ion.

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Recent progress of the PSI experiment on the Lamb shift in muonic hydrogen [1] has attracted interest to theory of the Lamb shift in light muonic atoms. Their study can provide us with information on certain nuclear structure effects with accuracy that is not available in any other experiment.

To obtain such data, one has to be able to separate quantum electrodynamics (QED) effects from the nuclear structure effects, and for this purpose an adequate QED theory providing high accuracy is required. Contributions to the energy levels in light muonic atoms and, in particular, to the Lamb shift fall into a few different well-distinguished classes. A specific theory stands behind each of them. There are corrections, the evaluation of which is identical for hydrogen and muonic hydrogen, and corrections that are specific for muonic atoms. The latter involve a certain part of QED, recoil effects and effects of the finite nuclear size. An important class of such specific contributions, which, in fact, also include the dominant term for the Lamb shift, is due to non-relativistic physics.

We remind that atomic momenta in light muonic atoms $\sim Z\alpha m_\mu \simeq 1.5Zm_e$ are compatible with the electron mass, while the atomic energy $\sim (Z\alpha)^2 m_\mu \simeq 0.01Z^2 m_e$ is much smaller than the electron mass. (The relativistic units in which $\hbar = c = 1$ are applied throughout the paper.) Such an environment produces an important sector of corrections, which deal with a non-relativistic bound muon, while the QED effects are present only

through the closed electron loops. Meanwhile the Compton wave length of electron $\lambda_e = 1/m_e$ determines the radius of the effective interaction induced by this kind of diagrams. The loops may be for either free-loop vacuum polarization (VP) effects, related to the Uehling, Källén-Sabry potential and higher-order diagrams, or the to light-by-light (LbL) scattering contributions.

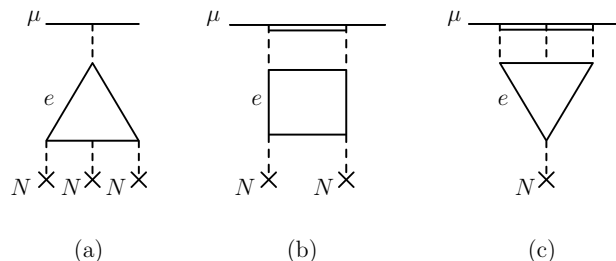


FIG. 1: Characteristic diagrams for three basic contributions of light-by-light scattering effects to the Lamb shift in muonic hydrogen in order $\alpha^5 m_\mu$. Here, N stands for a nucleus, which may be a proton, a deuteron etc. The horizontal double line is for the muon propagator in the Coulomb field.

The VP leading term is of order $\alpha(Z\alpha)^2 m_\mu$ and it has been known for a while, while the second-order VP contribution (of order $\alpha^2(Z\alpha)^2 m_\mu$) was calculated with appropriate accuracy for muonic hydrogen in [2] only relatively recently.

The accuracy of current and planned experiments requires a complete theory of non-relativistic contributions to the Lamb shift in order $\alpha^5 m_\mu$. The LbL contributions are depicted in Fig. 1, while the vacuum polarization diagrams are present in Fig. 2. Both kinds of contribu-

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tions have been known for a while, however, the results obtained up to now were only partial ones. In particular, in the case of muonic hydrogen the contribution in Fig. 1c has not yet been calculated, while there are also some questions [3] about applicability of the so-called scattering approximation applied in [4] to evaluate the contribution in Fig. 1b.

Certain LbL contributions have specific names. The first one in Fig. 1 is a so-called Wichmann-Kroll contribution and it was calculated for muonic hydrogen with sufficient accuracy in [3, 5]. It was also reproduced in [4]; we also confirm this contribution. For muonic deuterium and muonic helium-4 ion the results have been obtained in [6, 7] and we confirm the deuterium result [6] and obtain a result for a muonic helium ion

$$\Delta E_{1a}^{\text{He}} = -0.0198(4) \text{ meV} , \quad (1)$$

which is consistent with -0.02 meV of [7], but more accurate, and strongly disagrees with $+0.135 \text{ meV}$ of [8].

In our calculation we used approximations for the Wichmann-Kroll potential in the form

$$V_{\text{WK}}(r) = \frac{\alpha(Z\alpha)^2}{\pi} \frac{Z\alpha}{r} \times 0.361662331 \\ \times \exp \left[0.3728079x \right. \\ \left. - \sqrt{4.416798x^2 + 11.39911x + 2.906096} \right]$$

as discussed in [3] (see also [9]) and

$$V_{\text{WK}}(r) = \frac{(Z\alpha)^3 10^{-4}}{r} \begin{cases} \frac{1.528 - 0.489x}{1.374x^3 + 1.41x^2 + 2.672x + 1} , & x \leq 1 \\ \frac{0.207x^2 + 0.367x - 0.413}{x^6} , & x > 1 \end{cases}$$

as considered in [10] and [2]. Here $x = m_e r$. The results are consistent. Indeed, if higher accuracy is required, one can apply an exact expression [11] for $V_{\text{WK}}(r)$ as a two-dimensional integral.

The second term (Fig. 1b) is called ‘virtual-Delbrück-scattering contribution’. It has been calculated for muonic hydrogen in [4]. The calculation was based on [10, 12], where at first a scattering approximation was applied and subsequently a number of further approximations was made. We remind that the scattering approximation suggests that the external muon legs in the diagram in Fig. 1b are on-shell (i.e. $p^2 = m_\mu^2$) and the muon propagator there is substituted for a free one, i.e., the kinematics is exactly the same if one calculates a related Born scattering amplitude. Since atomic momenta $Z\alpha m_\mu$ in light muonic atoms are compatible with the electron mass m_e , the validity of such an approximation is questionable (see, e.g., the discussion in [3]).

We, however, have proved that the scattering approximation is applicable within the uncertainty of order $(Z\alpha)^2 m_\mu / m_e$ (in fractional units), which is at the level of about 1% in muonic hydrogen and deuterium and of about 4% in muonic helium. That is also correct

for other simplifying approximations, which were made in the calculations for this contribution in light muonic atoms [4, 6, 7, 10, 12]. A general idea of our evaluation is presented in Appendix A, while the details of our evaluation are to be published elsewhere [?].

Eventually, we conclude that the uncertainty of the method applied in [4, 6, 7] is substantially smaller than the uncertainty of the related numerical evaluations for the Lamb shift correction in muonic hydrogen [4], muonic deuterium [6] and muonic helium-4 ion [7].

The contribution in Fig. 1c has no specific name. Since any other LbL contribution has one (‘Wichmann-Kroll term’ and ‘virtual-Delbrück-scattering contribution’), sometimes it is referred to as a ‘light-by-light contribution’, which is somewhat confusing.

This contribution has remained uncalculated for a while. Studying applicability of the scattering approximation for the diagram in Fig. 1b, we have also managed to prove [?] that this remaining contribution can be expressed in terms of the well-known Wichmann-Kroll term

$$\Delta E_{1c} = \frac{1}{Z^2} \Delta E_{1a} . \quad (2)$$

The uncertainty of this identity is of order $(Z\alpha)^2 m_\mu / m_e$ (in fractional units), which is at the level of about 1% in muonic hydrogen and deuterium and of about 4% in muonic helium.

By combining our results on the uncertainty of various approximations with the numerical results of other authors, we obtain the complete result for all LbL contributions of Fig. 1. The result is listed in the summary table (Table I).

With identity (2) proved and a possibility to obtain a result for the Wichmann-Kroll (ΔE_{1a}) with high accuracy for any light muonic atom, the uncertainty in the calculation of the complete LbL contribution now comes from the virtual-Delbrück-scattering term, which should determine the eventual uncertainty of the non-relativistic $\alpha^5 m_\mu$ term for muonic hydrogen, deuterium and helium-ion.

Another major non-relativistic contribution in order $\alpha^5 m_\mu$ is due to vacuum polarization contributions. The VP terms of this order were studied for muonic hydrogen in [14]. The diagrams are depicted in Fig. 2, which includes contributions of the first (Fig 2a–d), second (Fig 2e, d) and the third (Fig 2f) order of non-relativistic perturbation theory (NRPT).

The most complicated terms are indeed related to the first line of Fig. 2, however, these diagrams were cross-checked due to their contributions to the anomalous magnetic moment of muon [15, 16] and we can rely on them.

The contributions of the second line of Fig. 2 are specific for muonic atoms and do not correlate directly with any calculation for the muon $g-2$. Those have been recalculated completely independently of [14], as well as part of diagrams in the first line.

We confirm the second-order terms of NRPT, while our result [17] for the third-order term (the last diagram

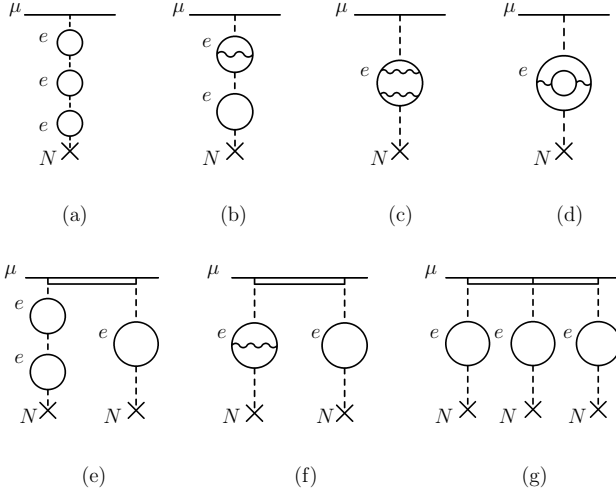


FIG. 2: Characteristic diagrams for free electron-VP contributions to the Lamb shift in muonic hydrogen in order $\alpha^5 m_\mu$.

in Fig. 2) disagrees with the one originally published in [14]. After a correction [18] their result agrees with ours. In our calculations we used techniques developed while investigating second-order vacuum-polarization contributions to the hyperfine structure of muonic hydrogen [19].

Term	$\Delta E^{\text{H}}(2p-2s)$ [meV]	$\Delta E^{\text{D}}(2p-2s)$ [meV]	$\Delta E^{\text{He}}(2p-2s)$ [meV]
1st order VP	205.007 36	227.634 67	1665.7729
2nd order VP	1.658 85	1.838 04	13.2769
3rd order VP	0.007 52	0.008 42(7)*	0.074(3)*
LbL (Fig. 1)	-0.000 71(15)*	-0.000 73(16)*	-0.005(10)*
NR total	206.673 02(15)	229.480 40(17)	1679.119(10)

TABLE I: The non-relativistic QED contributions to the Lamb shift $\Delta E(2s-2p)$ in light muonic atoms: hydrogen (H), deuterium (D), helium-4 ion (He), which includes VP contributions of the first ($\alpha(Z\alpha)^2 m_\mu$), the second ($\alpha^2(Z\alpha)^2 m_\mu$) and the third $\alpha^3(Z\alpha)^2 m_\mu$ (Fig. 2) order as well as a complete LbL contribution. The latter is a sum of contributions of order $\alpha(Z\alpha)^4 m_\mu$ (Fig. 1a), $\alpha^2(Z\alpha)^3 m_\mu$ (Fig. 1b) and $\alpha^3(Z\alpha)^2 m_\mu$ (Fig. 1c). Results marked with * are obtained in this work.

The diagrams in Fig. 2 were also discussed in various papers in the context of the Lamb shift in muonic deuterium [6] and muonic helium-4 ion [8]. For this purpose a part of the contributions was recalculated there.

Here, we reevaluated all the vacuum-polarization contributions and the results are listed in Table I. Comparing with the paper mentioned we have to acknowledge that our results are not in complete agreement with theirs.

In [6] only contributions of Fig. 2a and b were directly calculated for muonic deuterium and the results for them agree with ours. However, that result were not applied there but instead the muonic-hydrogen result was ‘re-scaled’. That was achieved by assuming that the result

for muonic hydrogen, presented in [14] in the form of

$$\Delta E(2p-2s) = C_3 \left(\frac{\alpha}{\pi}\right)^3 (Z\alpha)^2 m_r, \quad (3)$$

where m_r is the muon reduced mass, can be directly applied for the muonic deuterium. That has not been claimed in [14] and is indeed incorrect and the value of the coefficient [17, 18]

$$C_3 = 0.118\,680(12) \quad (4)$$

is valid only for muonic hydrogen (cf. with $C_3 = 0.120\,045(12)$ from [14], which needs a correction [17] as explained above). The related values for other light muonic atoms, obtained here,

$$C_3 = \begin{cases} 0.1262(11), & \text{for } \mu\text{D}, \\ 0.270(17), & \text{for } \mu\text{He}^+. \end{cases} \quad (5)$$

obviously differ from (4).

The muonic-helium-4 paper [8] lacks a complete result and only a part of diagrams of Fig. 2 were recalculated. Our results are not in fair agreement, and in particular we strongly disagree in the contribution of Fig. 2e for muonic helium. In the Wichmann-Kroll contribution (Fig. 1a) we also strongly disagree with the result [8], while we agree with the result [7], for which we obtain higher accuracy (see (1)).

Finally, we summarize in Table I a complete theory of non-relativistic QED contributions to the Lamb shift in muonic hydrogen, deuterium and helium ion up to the order $\alpha^5 m_\mu$ (see [3] for references to the calculation of the low-order corrections).

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Note Added: After submission of this work, we have calculated the LbL contribution within a static-muon approximation (see Appendix A). The preliminary results for ΔE_{LbL} , which are 0.001 15(1) meV for μH , 0.001 24(1) meV for μD , and 0.0114(4) meV for μHe^+ , are somewhat below the former results [4, 6, 7], but still are in fair agreement with them. These more accurate results will be reported in detail in a future publication [13].

Appendix A: On approximations for the LbL contribution

We have proven a kind of theorem [?] that the diagrams in Fig. 1 can be calculated in light muonic

atoms (for simplicity we consider further muonic hydrogen) within the static-muon approximation, in which the complete muon-line factor shrinks to

$$\mathcal{F}(\mathbf{q}) = \int \frac{d^3p}{(2\pi)^3} \Psi^*(\mathbf{p}) \Psi(\mathbf{p} + \mathbf{q}),$$

where \mathbf{q} is the total momentum transfer to the muon line, Ψ is the wave function and the error is of the order of $(Z\alpha)^2 m_\mu/m_e$.

The scattering approximation [4, 6] agrees with the static-muon approximation within the same uncertainty.

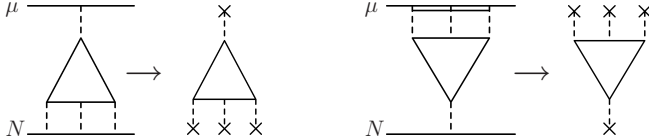


FIG. 3: Reduction of diagrams with free/bound fermion propagators to static-fermion diagrams (i.e. to the static-proton approximation (= the external-field approximation) (left) and to the static-muon approximation (right).

The proof of the ‘theorem’ is to be presented elsewhere [?] and here we explain its main idea. The proton and muon are both non-relativistic (NR) particles. Their descriptions are very similar.

Let us first compare the related diagrams with free NR fermion propagators (cf. Fig. 3). The expressions for the muon and proton lines are identical. Apparently, one can approximate the proton line within the external field approach or, which is the same, by a static proton. To transform the complete NR expressions to the static-proton case, we have to neglect the proton kinetic energy in the proton propagators. Once that is done, after a chain of identities we should arrive at the external-field approximation.

A reason to neglect the energy is the fact that a characteristic energy, related to a particle of the mass M , is $E_M \sim \gamma^2/M$, where $\gamma = Z\alpha m_\mu$ is a characteristic atomic momentum. One can prove that we can expand using small parameters E_M/m_e and E_M/γ . However, in muonic hydrogen they are of the same order since $\gamma \sim m_e$. The parameter E_M/m_e (and the related error) differs indeed for a muon and proton, but it is small for both.

Thus as long as the muon propagator is a free one there is no difference in proving that we can apply the static-muon approximation and the static-proton approximation (see Fig. 3).

Meantime, in reality the situation for a muon and a proton is somewhat different. The muon characteristic momentum is of the same order as the electron mass $\gamma \sim m_e$, and we should treat it as a bound one.

The NR Coulomb Green function of a muon includes

$$G_C(E, \mathbf{p}, \mathbf{p}') = i \sum_{\lambda} \frac{|\lambda(\mathbf{p})\rangle \langle \lambda(\mathbf{p}')|}{E - E_{\lambda} + i0},$$

a summation over all intermediate states λ of continuous and discrete spectrum, involving energy of the intermediates. The characteristic energy of an intermediate state is indeed of order of magnitude of the atomic bound energy $E_{\lambda} \sim \gamma^2/m_\mu$ and we can neglect it, as we already did in the case of the free propagators. After that, the sum over intermediates shrinks to the unity operator and the Coulomb propagator becomes equal to a free one with the kinetic-energy term $\mathbf{p}^2/2m_\mu$ neglected

$$i \sum_{\lambda} \frac{|\lambda\rangle \langle \lambda|}{E - E_{\lambda} + i0} \rightarrow i \sum_{\lambda} \frac{|\lambda\rangle \langle \lambda|}{E + i0} = \frac{i}{E + i0}.$$

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